

# Schwarzschild horizon dynamics and $SU(2)$ Chern-Simons theory

Romesh K. Kaul<sup>1,\*</sup> and Parthasarathi Majumdar<sup>2,†</sup>

<sup>1</sup>*The Institute of Mathematical Sciences,  
CIT Campus, Chennai-600 113, INDIA.*

<sup>2</sup>*Saha Institute of Nuclear Physics,  
AF/1 Bidhannagar, Kolkata 700 064, INDIA.*

## Abstract

We discuss the effect of different choices in partial gauge fixing of bulk local Lorentz invariance, on the description of the horizon degrees of freedom of a Schwarzschild black hole as an  $SU(2)$  Chern-Simons theory with specific sources. A classically equivalent description in terms of an  $ISO(2)$  Chern-Simons theory is also discussed. Further, we demonstrate that both these descriptions can be partially gauge fixed to a horizon theory with  $U(1)$  local gauge invariance, with the solder form sources being subject to extra constraints in directions orthogonal to an internal vector field left invariant by  $U(1)$  transformations. Seemingly disparate approaches on characterization of the horizon theory for the Schwarzschild black hole (as well as spherical Isolated Horizons in general) are thus shown to be equivalent physically.

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\*Electronic address: [kaul@imsc.res.in](mailto:kaul@imsc.res.in)

†Electronic address: [parthasarathi.majumdar@saha.ac.in](mailto:parthasarathi.majumdar@saha.ac.in)

## I. INTRODUCTION

The event horizon (EH) of a black hole spacetime (and more generally an Isolated Horizon (IH)) [1]), is a null inner boundary of the part of the entire spacetime manifold accessible to asymptotic observers. It has the topology of  $R \times S^2$  and a *degenerate* intrinsic three-metric. Because of this latter property, it is not possible to describe the horizon degrees of freedom in terms of a Lagrange density with standard kinetic terms where contractions are usually made with the inverse metric. In this sense, the horizon three-fold does not support any local propagating degree of freedom. The only possible degrees of freedom on the horizon have to be global or topological, described by a *topological* (metric independent) quantum field theory. Three dimensional Chern-Simons theories appear to be good candidate topological field theories for this description.

In Loop Quantum Gravity (LQG), bulk spacetime properties are described in terms of the Barbero-Immirzi family of  $SU(2)$  connections [2] obtained from a partially gauged fixed  $SO(1,3)$  theory. All physics associated with bulk spacetime geometry must be invariant under local  $SU(2)$  transformations. Since, at the classical level, the degrees of freedom and their dynamics on an EH (IH) are *completely determined by the geometry and dynamics in the bulk*, the theory of the horizon degrees of freedom, has to imbibe this  $SU(2)$  gauge invariance from the bulk. This implies that the horizon degrees of freedom should be described by a topological  $SU(2)$  Chern-Simons theory on the three-manifold  $R \times S^2$ , coupled to appropriate sources derived from tetrad components in the bulk. However, there are ambiguities in partially gauge fixing the bulk local Lorentz invariance to  $SU(2)$ . Studying the effect of these on the horizon theory is the main thrust of this paper.

Use of  $SU(2)$  gauge theory to count the microstates associated with a two-dimensional surface has a long history. Inspired by the proposal of Crane that quantum gravity be described by a topological field theory [3] and the holographic hypothesis of 't Hooft and Susskind [4], it was Smolin who first explored the use of  $SU(2)$  Chern-Simons theory induced on boundary satisfying self-dual boundary conditions in Euclidean gravity and also demonstrated that such a boundary theory obeys the Bekenstein bound [5]. This was followed by the work of Krasnov who applied these ideas to the black hole horizon and used the ensemble of quantum states of  $SU(2)$  Chern-Simons theory associated with the spin assignments of the punctures on the surface to count the microstates, leading to an area law

for the entropy [6]. The coupling of the Chern-Simons theory was argued to be proportional to the horizon area and also inversely proportional to the Barbero-Immirzi parameter  $\gamma$ . This was the first application of  $SU(2)$  Chern-Simons theory for calculating the black hole entropy. On the other hand, within the Loop Quantum Gravity, assuming that the geometry of the fluctuating black hole horizon is given by the quantum states associated with the intersections of knots carrying  $SU(2)$  spins impinging on the two-dimensional surface, a counting procedure was developed by Rovelli, again obtaining an area law for the entropy [7]. In the general context of Isolated Horizons, application of  $SU(2)$  Chern-Simons theory as a boundary theory came with the work of Ashtekar, Baez, Corichi and Krasnov [8] and was further developed in ref.s. [9–12].

Following the derivation of the area law for the entropy of large area IHs in [8], corrections due to quantum spacetime fluctuations, leading logarithmic in area  $-\frac{3}{2} \ln A$  (with this definite coefficient  $-3/2$ ) and subleading in inverse powers of area, were obtained within the framework of this  $SU(2)$  Chern-Simons theory in [10]. These were done in the approximation where spin  $1/2$  representations were placed on the punctures of the spatial slice  $S^2$  of the horizon. Such configurations provide the dominant contribution to the dimensionality of the IH Hilbert space. The coefficient of the leading area term depends on the Barbero-Immirzi parameter  $\gamma$ . Matching this with the Bekenstein-Hawking area law fixes a definite value of  $\gamma$ . In fact, the logarithmic corrections to the area law, obtained in this framework, are the first ever *signature* corrections thrown up by quantization of IHs within LQG, obtained by *using Chern-Simons theories*. That these logarithmic corrections do not depend on the value of  $\gamma$  also emerges from these studies. An improvement over the approximation used in these calculations has been achieved by including the contributions of spins other than  $1/2$  on some of the punctures [14], which changes the coefficient of the leading area term and thus improves the value of  $\gamma$  by about 10%. In these counting schemes, however, the logarithmic correction,  $-\frac{3}{2} \ln A$ , which does not depend on  $\gamma$ , is unaffected. In fact this leading  $\log(\text{area})$  correction is rather generally insensitive to the value of the spins placed on the punctures. For example, it has been explicitly shown that placing spin 1 representations on all the punctures changes the value of  $\gamma$ , but leaves the coefficient of the leading logarithmic correction unchanged [12].

Recently, there has been a resurgent interest in this  $SU(2)$  Chern-Simons theoretic description of Isolated Horizons started by [15] and followed by others [16–19]. Some of these

papers have recalculated and confirmed the nature of the leading logarithmic correction to the Bekenstein-Hawking area law for microcanonical entropy of isolated horizons, with the definite coefficient  $-3/2$ , found a decade earlier in [10, 11]. However, in these latter formulations, the coupling strength of the Chern Simons mysteriously appears to diverge for a value of the Barbero-Immirzi parameter which seems to have no particular significance.

The  $SU(2)$  Chern-Simons description of the horizon degrees of freedom has occasionally been viewed in the literature as a *counterpoint* to the description in terms of a  $U(1)$  Chern-Simons theory [1, 8]. These, apparently disparate, points of view, are in fact quite reconcilable. The result follows from the fact that what is relevant in the problem on hand are properties of fields on the spatial slice  $S^2$  of the horizon. It is indeed always possible to partially gauge fix the  $SU(2)$  theory on  $S^2$  to a theory with only a left over  $U(1)$  invariance. In particular, as argued in [13], to go from the  $SU(2)$  theory to the  $U(1)$  theory in the gauge fixed formulation, there are additional constraints for the solder forms orthogonal to the direction specified by an internal space unit vector left invariant by a  $U(1)$  subgroup of  $SU(2)$  gauge group. These constraints only reflect the  $SU(2)$  underpinnings of the  $U(1)$  theory.

This special property of being able to fix an  $SU(2)$  gauge invariance to a  $U(1)$  gauge invariance, obtains only on  $S^2$ . One direct way of seeing this is as follows: In an  $SU(2)$  gauge theory described through the triplet of field strength  $F_{\theta\phi}^{(i)}$  ( $i = 1, 2, 3$ ) on  $S^2$ , we can always rotate the field strength through an  $SU(2)$  gauge transformation to have only one nonzero component lying in the direction preserved by a subgroup  $U(1)$ :  $F_{\theta\phi}^{(i)} \rightarrow F_{\theta\phi}'^{(i)} = (F_{\theta\phi}'^{(1)}, 0, 0)$ . Next, for such a field strength on  $S^2$ , the antisymmetric two-tensor  $F_{\theta\phi}'^{(1)}$  is given by the curl of a vector field with components  $A_\theta, A_\phi$ :  $F_{\theta\phi}'^{(1)} = \partial_\theta A_\phi - \partial_\phi A_\theta$ , which defines the  $U(1)$  curvature. Clearly, this gauge fixing implies that  $SU(2)$  and  $U(1)$  gauge fields on  $S^2$  have the *same physical content*.

In the present paper, we revisit the  $SU(2)$  Chern-Simons description of the Schwarzschild event horizon, and discuss effects on the horizon induced by various ways of partial gauge fixing of bulk local Lorentz invariance. In particular, the mysteriously diverging Chern-Simons coupling found recently in [15] is seen to emerge straightforwardly as a consequence of this gauge fixing. An equivalent description in terms of an  $ISO(2)$  Chern-Simons theory is also discussed in this context. How an effective Chern-Simons theory, with these higher gauge invariances all gauge fixed to a  $U(1)$ , ties up these approaches, with appropriate

constraints corresponding to the gauge fixing, is explained in some detail. These results emerge within a discussion of properties of the future EH of the Kruskal-Szekeres extension of the Schwarzschild spacetime, but generalize to any spherical Isolated Horizon. Dealing with an exact black hole solution allows us to extract information on its horizon dynamics in a manner that is physically equivalent to extant approaches based on the Hamiltonian analysis of isolated horizons [1], at least for this particular solution. Is this generic enough for all spherical isolated horizons? In what follows, we point out the precise features in our results which also emerge in the general case of spherical isolated horizons obtained in earlier work cited above.

In Section II, we display an appropriate set of the tetrad components and corresponding spin connection components. We explicitly exhibit the gauge equivalent class of tetrads in terms of a single function  $\alpha(x)$  which is the sole ambiguity in the choice of a local *Lorentz frame* for the Schwarzschild metric. Other tetrad sets are related to ours only through different choices of *coordinates*. On the black hole (future) horizon, in Section III, the field strength components associated with our connection fields are shown to satisfy a set of equations which exhibit a left over invariance under  $U(1)$  gauge transformations, when the scale function assumes certain specific values. We shall then further demonstrate that, for other local Lorentz frame choices in the bulk, these horizon equations can also be interpreted as a gauge fixed version of an  $ISO(2)$  Chern-Simons theory. In addition, there is, for different local Lorentz frame choices, an alternative description in terms of a Chern-Simons theory of the Barbero-Immirzi  $SU(2)$  gauge fields which in a gauge fixed version reproduces the  $U(1)$  gauge theory. This will be presented in Section IV. In this gauge fixed  $U(1)$  formulation, the sources in the direction orthogonal to  $U(1)$  subgroup are constrained to vanish. In particular, as emphasized in the earlier analysis in [13], the two components of  $SU(2)$  triplet solder forms on the spatial slice of the horizon orthogonal to the direction specified by the  $U(1)$  subgroup are indeed zero as they should be. Finally, Section V will contain a few concluding remarks.

While our analysis presented here is for the future (black hole) horizon of the Kruskal-Szekeres extended Schwarzschild spacetime, rather than the past (white hole) horizon, similar conclusions would ensue for that case as well.

## II. SCHWARZSCHILD METRIC IN KRUSKAL-SZEKERES COORDINATES

The Schwarzschild metric, expressed in the Kruskal-Szekeres null coordinates  $v$  and  $w$ , is:

$$ds^2 = -2A(r) dv dw + r^2(v, w) (d\theta^2 + \sin^2 \theta d\phi^2), \quad A(r) = \frac{4r_0^3}{r} \exp\left(-\frac{r}{r_0}\right) \quad (1)$$

where  $r$  is given implicitly by:

$$-2vw = \left(\frac{r}{r_0} - 1\right) \exp\left(\frac{r}{r_0}\right) \quad (2)$$

The exterior region ( $r > r_0$ ) of the black hole is given by:  $vw < 0$ ,  $v > 0$ ,  $w < 0$ . The interior region ( $r_0 > r > 0$ ) is:  $0 < 2vw < 1$ ,  $v > 0$ ,  $w > 0$ . In terms of these coordinates, the past and future event horizons are given by:  $vw = 0$ . The outgoing null geodesics are given by  $w = \text{constant}$  and the ingoing null geodesics by  $v = \text{constant}$ . The curvature singularity ( $r = 0$ ) is described by  $2vw = 1$ .

Corresponding to the metric (1), non-zero Christoffel symbols  $\Gamma_{\mu\nu}^\lambda$  are:

$$\begin{aligned} \Gamma_{vv}^v &= \partial_v \ln A, & \Gamma_{\theta\theta}^v &= \frac{r^2}{A} \partial_w \ln r, & \Gamma_{\phi\phi}^v &= \frac{r^2 \sin^2 \theta}{A} \partial_w \ln r; \\ \Gamma_{ww}^w &= \partial_w \ln A, & \Gamma_{\theta\theta}^w &= \frac{r^2}{A} \partial_v \ln r, & \Gamma_{\phi\phi}^w &= \frac{r^2 \sin^2 \theta}{A} \partial_v \ln r; \\ \Gamma_{v\theta}^\theta &= \partial_v \ln r, & \Gamma_{w\theta}^\theta &= \partial_w \ln r, & \Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta; \\ \Gamma_{v\phi}^\phi &= \partial_v \ln r, & \Gamma_{w\phi}^\phi &= \partial_w \ln r, & \Gamma_{\theta\phi}^\phi &= \cot \theta \end{aligned} \quad (3)$$

Now we choose an appropriate set of tetrad fields which are compatible with the metric (1). In the following, we shall restrict ourselves only to the exterior region of the black hole ( $v > 0$ ,  $w < 0$ ). In this region, we take the tetrad fields as:

$$\begin{aligned} e_\mu^0 &= \sqrt{\frac{A}{2}} \left( \frac{w}{\alpha} \partial_\mu v + \frac{\alpha}{w} \partial_\mu w \right), & e_\mu^1 &= \sqrt{\frac{A}{2}} \left( \frac{w}{\alpha} \partial_\mu v - \frac{\alpha}{w} \partial_\mu w \right), \\ e_\mu^2 &= r \partial_\mu \theta, & e_\mu^3 &= r \sin \theta \partial_\mu \phi \end{aligned} \quad (4)$$

Here  $\alpha$  is an arbitrary function of the coordinates; every choice of  $\alpha(x)$  characterizes the local Lorentz frame in the indefinite metric plane  $\mathcal{I}$  of the Schwarzschild spacetime whose spherical symmetry implies that it has the topology  $\mathcal{I} \otimes S^2$ . Corresponding to these tetrad

fields, the spin connections satisfying the relation  $\partial_\mu e_\nu^I - \Gamma_{\mu\nu}^\lambda e_\lambda^I + \omega_\mu^I{}_J e_\nu^J = 0$  are

$$\begin{aligned}\omega_\mu^{01} &= -\frac{1}{2} \left(1 - \frac{r_0^2}{r^2}\right) \frac{1}{v} \partial_\mu v - \frac{1}{2} \left(1 + \frac{r_0^2}{r^2}\right) \frac{1}{w} \partial_\mu w + \partial_\mu \ln \alpha, & \omega_\mu^{23} &= -\cos \theta \partial_\mu \phi \\ \omega_\mu^{02} &= -\sqrt{\frac{A}{2}} \frac{1}{2r_0} \left(\frac{vw}{\alpha} + \alpha\right) \partial_\mu \theta, & \omega_\mu^{03} &= -\sqrt{\frac{A}{2}} \frac{\sin \theta}{2r_0} \left(\frac{vw}{\alpha} + \alpha\right) \partial_\mu \phi \\ \omega_\mu^{12} &= -\sqrt{\frac{A}{2}} \frac{1}{2r_0} \left(\frac{vw}{\alpha} - \alpha\right) \partial_\mu \theta, & \omega_\mu^{13} &= -\sqrt{\frac{A}{2}} \frac{\sin \theta}{2r_0} \left(\frac{vw}{\alpha} - \alpha\right) \partial_\mu \phi\end{aligned}\quad (5)$$

The curvature tensor  $R_{\mu\nu}{}^{IJ} = \partial_\mu \omega_\nu^{IJ} - \partial_\nu \omega_\mu^{IJ} + \omega_\mu^{IK} \omega_{\nu K}^J - \omega_\nu^{IK} \omega_{\mu K}^J$  for the spin connections (5) is given by:

$$\begin{aligned}R_{\mu\nu}{}^{01} &= \frac{2r_0}{r^3} \Sigma_{\mu\nu}^{01}, & R_{\mu\nu}{}^{02} &= -\frac{r_0}{r^3} \Sigma_{\mu\nu}^{02}, & R_{\mu\nu}{}^{03} &= -\frac{r_0}{r^3} \Sigma_{\mu\nu}^{03}, \\ R_{\mu\nu}{}^{23} &= \frac{2r_0}{r^3} \Sigma_{\mu\nu}^{23}, & R_{\mu\nu}{}^{31} &= -\frac{r_0}{r^3} \Sigma_{\mu\nu}^{31}, & R_{\mu\nu}{}^{12} &= -\frac{r_0}{r^3} \Sigma_{\mu\nu}^{12}\end{aligned}\quad (6)$$

where the solder forms  $\Sigma_{\mu\nu}^{IJ} = e_{[\mu}^I e_{\nu]}^J \equiv \frac{1}{2} (e_\mu^I e_\nu^J - e_\nu^I e_\mu^J)$ , in the exterior region ( $v > 0$ ,  $w < 0$ ), are

$$\begin{aligned}\Sigma_{\mu\nu}^{01} &= -A \partial_{[\mu} v \partial_{\nu]} w, & \Sigma_{\mu\nu}^{23} &= r^2 \sin \theta \partial_{[\mu} \theta \partial_{\nu]} \phi \\ \Sigma_{\mu\nu}^{02} &= r \sqrt{\frac{A}{2}} \left( \frac{w}{\alpha} \partial_{[\mu} v \partial_{\nu]} \theta + \frac{\alpha}{w} \partial_{[\mu} w \partial_{\nu]} \theta \right), \\ \Sigma_{\mu\nu}^{03} &= r \sin \theta \sqrt{\frac{A}{2}} \left( \frac{w}{\alpha} \partial_{[\mu} v \partial_{\nu]} \phi + \frac{\alpha}{w} \partial_{[\mu} w \partial_{\nu]} \phi \right), \\ \Sigma_{\mu\nu}^{12} &= r \sqrt{\frac{A}{2}} \left( \frac{w}{\alpha} \partial_{[\mu} v \partial_{\nu]} \theta - \frac{\alpha}{w} \partial_{[\mu} w \partial_{\nu]} \theta \right) \\ \Sigma_{\mu\nu}^{31} &= -r \sin \theta \sqrt{\frac{A}{2}} \left( \frac{w}{\alpha} \partial_{[\mu} v \partial_{\nu]} \phi - \frac{\alpha}{w} \partial_{[\mu} w \partial_{\nu]} \phi \right)\end{aligned}\quad (7)$$

Next, LQG is described in terms of Barbero-Immirzi  $SU(2)$  gauge fields [2] which are linear combinations of the the connection components involving the Barbero-Immirzi parameter  $\gamma$ . To make contact with this, we introduce the  $SU(2)$  gauge field:

$$A_\mu^{(i)} = \gamma \omega_\mu^{0i} - \frac{1}{2} \epsilon^{ijk} \omega_\mu^{jk} \quad (8)$$

Substituting for the spin connections from (5), this yields in the exterior region ( $v > 0$ ,  $w < 0$ ):

$$\begin{aligned}A_\mu^{(1)} &= \gamma \omega_\mu^{01} - \omega_\mu^{23} = -\frac{\gamma}{2} \left[ \left(1 - \frac{r_0^2}{r^2}\right) \frac{1}{v} \partial_\mu v + \left(1 + \frac{r_0^2}{r^2}\right) \frac{1}{w} \partial_\mu w - 2\partial_\mu \ln \alpha \right] + \cos \theta \partial_\mu \phi \\ A_\mu^{(2)} &= \gamma \omega_\mu^{02} - \omega_\mu^{31} = -\sqrt{\frac{A}{2}} \frac{1}{2r_0} \left[ \gamma \left(\frac{vw}{\alpha} + \alpha\right) \partial_\mu \theta + \sin \theta \left(\frac{vw}{\alpha} - \alpha\right) \partial_\mu \phi \right], \\ A_\mu^{(3)} &= \gamma \omega_\mu^{03} - \omega_\mu^{12} = -\sqrt{\frac{A}{2}} \frac{1}{2r_0} \left[ \gamma \sin \theta \left(\frac{vw}{\alpha} + \alpha\right) \partial_\mu \phi - \left(\frac{vw}{\alpha} - \alpha\right) \partial_\mu \theta \right]\end{aligned}\quad (9)$$

The choice of the tetrad fields as in eqn.(4) is not unique; we could have used any other choice compatible with the metric (1).

Now let us restrict our discussion to the event horizon by taking the limit to the horizon from the exterior region to unravel the properties of the various fields on the horizon.

### III. BLACK HOLE HORIZON AND $ISO(2)$ CHERN-SIMONS THEORY

The black hole horizon is the future horizon given by  $w = 0$ , which is a null three-manifold  $\Delta$ , topologically  $R \times S^2$ , spanned by the coordinates  $a = (v, \theta, \phi)$  where  $0 < v < \infty$ ,  $0 \leq \theta < \pi$ ,  $0 \leq \phi < 2\pi$ . The evolution parameter is  $v$ . The null and future directed geodesics given by  $v = \text{constant}$  are infalling into  $\Delta$ . The foliation of the manifold  $\Delta$  is provided by  $v = \text{constant}$  surfaces, each an  $S^2$ .

The relevant tetrad fields  $e_a^I$  from (4) on  $\Delta$  are given by:  $e_a^0 \hat{=} 0$ ,  $e_a^1 \hat{=} 0$ ,  $e_a^2 \hat{=} r_0 \partial_a \theta$ ,  $e_a^3 \hat{=} r_0 \sin \theta \partial_a \phi$  where  $a = (v, \theta, \phi)$  (we denote equalities on  $\Delta$ , that is for  $w = 0$ , by the symbol  $\hat{=}$ ). The intrinsic metric on  $\Delta$  is:  $q_{ab} = e_a^I e_{Ib} \hat{=} m_a \bar{m}_b + m_b \bar{m}_a$  with  $m_a \equiv \frac{r_0}{\sqrt{2}} (\partial_a \theta + i \sin \theta \partial_a \phi)$ . This metric is indeed degenerate with its signature  $(0, +, +)$ .

Notice that the solder fields on the horizon  $\Delta$  are:

$$\text{all } \Sigma_{ab}^{IJ} \hat{=} 0 \quad \text{except } \Sigma_{ab}^{23} \hat{=} r_0^2 \sin \theta \partial_{[a} \theta \partial_{b]} \phi \quad (10)$$

and the spin connection fields are:

$$\begin{aligned} \omega_a^{01} &\hat{=} \frac{1}{2} \partial_a \ln \beta, & \omega_a^{23} &= -\cos \theta \partial_a \phi, \\ \omega_a^{02} &\hat{=} -\sqrt{\beta} \partial_a \theta, & \omega_a^{03} &\hat{=} -\sqrt{\beta} \sin \theta \partial_a \phi \\ \omega_a^{12} &\hat{=} \sqrt{\beta} \partial_a \theta, & \omega_a^{13} &\hat{=} \sqrt{\beta} \sin \theta \partial_a \phi \end{aligned} \quad (11)$$

where  $\beta \equiv \frac{\alpha^2}{2e}$  and we have used  $A(r_0) = \frac{4r_0^2}{e}$  with  $e \equiv \exp(1)$ . The corresponding curvature tensor components are:

$$\text{all } R_{ab}^{IJ}(\omega) \hat{=} 0 \quad \text{except } R_{ab}^{23}(\omega) \hat{=} 2 \sin \theta \partial_{[a} \theta \partial_{b]} \phi = \frac{2}{r_0^2} \Sigma_{ab}^{23} \equiv \frac{2\gamma}{r_0^2} \Sigma_{ab}^{(1)} \quad (12)$$

where we have introduced  $\Sigma_{ab}^{(1)} = \gamma^{-1} \Sigma_{ab}^{23}$ . These equations can be interpreted as a  $U(1)$  Chern-Simons theory with  $\omega_a^{23}$  as the  $U(1)$  gauge field.



Notice, the connection component  $\omega_a^{01}$  in (11) is *pure gauge* and hence can be rotated away to zero by a boost gauge transformation  $\omega_a^{IJ} \rightarrow \omega_a'^{IJ}$  where:

$$\begin{aligned}\omega_a'^{01} &= \omega_a^{01} - \partial_a \xi, & \omega_a'^{23} &= \omega_a^{23} \\ \omega_a'^{02} &= \cosh \xi \omega_a^{02} + \sinh \xi \omega_a^{12}, & \omega_a'^{03} &= \cosh \xi \omega_a^{03} + \sinh \xi \omega_a^{13}, \\ \omega_a'^{12} &= \sinh \xi \omega_a^{02} + \cosh \xi \omega_a^{12}, & \omega_a'^{13} &= \sinh \xi \omega_a^{03} + \cosh \xi \omega_a^{13}\end{aligned}$$

From these, if we choose  $\xi = \frac{1}{2} \ln(\frac{2\beta}{c})$  where  $c$  is independent of the coordinates  $v, \theta, \phi$ , the connections fields (11) then transform to:

$$\begin{aligned}\omega_a'^{01} &\hat{=} 0 & \omega_a'^{23} &= -\cos \theta \partial_a \phi, \\ \omega_a'^{02} &\hat{=} -\frac{c}{\sqrt{2}} \partial_a \theta, & \omega_a'^{03} &\hat{=} -\frac{c}{\sqrt{2}} \sin \theta \partial_a \phi \\ \omega_a'^{12} &\hat{=} \frac{c}{\sqrt{2}} \partial_a \theta, & \omega_a'^{13} &\hat{=} \frac{c}{\sqrt{2}} \sin \theta \partial_a \phi\end{aligned}\tag{13}$$

These are really the gauge fields of  $ISO(2)$  theory. To see this explicitly, we rewrite the fields as the following combinations:

$$\begin{aligned}\mathcal{A}_a^1 &\equiv \omega_a'^{23} \hat{=} -\cos \theta \partial_a \phi, \\ \mathcal{A}_a^2 &\equiv \frac{1}{\sqrt{2}} (\omega_a'^{02} - \omega_a'^{12}) \hat{=} -c \partial_a \theta, \\ \mathcal{A}_a^3 &\equiv \frac{1}{\sqrt{2}} (\omega_a'^{03} + \omega_a'^{13}) \hat{=} -c \sin \theta \partial_a \phi\end{aligned}\tag{14}$$

and

$$\tilde{\mathcal{A}}_a^2 \equiv \frac{1}{\sqrt{2}} (\omega_a'^{02} + \omega_a'^{12}) \hat{=} 0, \quad \tilde{\mathcal{A}}_a^3 \equiv \frac{1}{\sqrt{2}} (\omega_a'^{03} - \omega_a'^{13}) \hat{=} 0, \quad \mathcal{A}_a^4 \equiv \omega_a'^{01} \hat{=} 0\tag{15}$$

The fields  $(\mathcal{A}_a^1, \mathcal{A}_a^2, \mathcal{A}_a^3)$  can be readily recognized as the three gauge fields of  $ISO(2)$  subgroup. The three generators of  $ISO(2)$  subgroup are given in terms of the generators of the Lorentz algebra  $M_{IJ}$  by:  $\mathcal{P} = \frac{1}{\sqrt{2}}(K_2 - J_3)$ ,  $\mathcal{Q} = \frac{1}{\sqrt{2}}(K_3 + J_2)$ , and  $\mathcal{J} = J_1$ . where  $K_i \equiv M_{0i} = -M^{0i}$ ,  $J_i \equiv \frac{1}{2} \epsilon_{ijk} M_{jk}$ . These satisfy the algebra:  $[\mathcal{P}, \mathcal{Q}] = 0$ ,  $[\mathcal{J}, \mathcal{P}] = \mathcal{Q}$ ,  $[\mathcal{J}, \mathcal{Q}] = -\mathcal{P}$ . This is the subgroup of Lorentz transformations that leave a null internal vector invariant. For Schwarzschild spacetime, this null vector is the Killing vector corresponding to the timelike isometry of the exterior metric; on the horizon, this Killing vector turns null. The  $ISO(2)$  transformations correspond to the subgroup of local Lorentz transformations which leave this vector invariant on the horizon [20].

For the  $ISO(2)$  theory for the gauge fields (14), the field strength components satisfy the relations:

$$\begin{aligned}\mathcal{F}_{ab}^1 &\equiv 2 \partial_{[a} \mathcal{A}_{b]}^1 \hat{=} 2 \sin \theta \partial_{[a} \theta \partial_{b]} \phi \hat{=} \frac{2\gamma}{r_0^2} \Sigma_{ab}^{(1)} , \\ \mathcal{F}_{ab}^2 &\equiv 2 \partial_{[a} \mathcal{A}_{b]}^2 + 2 \mathcal{A}_{[a}^1 \mathcal{A}_{b]}^3 \hat{=} 0 , \\ \mathcal{F}_{ab}^3 &\equiv 2 \partial_{[a} \mathcal{A}_{b]}^3 - 2 \mathcal{A}_{[a}^1 \mathcal{A}_{b]}^2 \hat{=} 0\end{aligned}\quad (16)$$

These equations are invariant under the  $U(1)$  subgroup of  $ISO(2)$  gauge transformations. Hence these represent the equations of motion of an  $ISO(2)$  Chern-Simons theory gauge fixed to  $U(1)$  with source  $\Sigma_{\theta\phi}^{(1)}$  in the direction of the  $U(1)$  subgroup and coupling  $k = \frac{\pi r_0^2}{\gamma}$ . To see that this indeed is the case consider the equations of motion of the  $ISO(2)$  Chern-Simons theory of gauge fields  $\mathcal{A}_a^i$  and their field strength  $\mathcal{F}_{ab}^i(\mathcal{A}')$  with coupling constant  $k$  and a specific source given by:

$$\mathcal{F}_{v\theta}^i(\mathcal{A}') = 0 , \quad \mathcal{F}_{v\phi}^i(\mathcal{A}') = 0 , \quad \frac{k}{2\pi} \mathcal{F}_{\theta\phi}^i(\mathcal{A}') = J^i \quad (17)$$

These equations are covariant under the  $ISO(2)$  gauge transformations which consist of two sets: (a) The  $U(1)$  transformations, associated with the generators  $T_1 \equiv -\mathcal{J}$ , on the gauge fields:

$$\mathcal{A}_a'^1 \rightarrow \mathcal{A}_a'^1 - \partial_a \alpha , \quad \mathcal{A}_a'^2 \rightarrow \cos \alpha \mathcal{A}_a'^2 + \sin \alpha \mathcal{A}_a'^3 , \quad \mathcal{A}_a'^3 \rightarrow -\sin \alpha \mathcal{A}_a'^2 + \cos \alpha \mathcal{A}_a'^3 \quad (18)$$

where  $\alpha$  is the local transformation parameter. The field strength components change as:

$$\mathcal{F}_{ab}'^1 \rightarrow \mathcal{F}_{ab}'^1 , \quad \mathcal{F}_{ab}'^2 \rightarrow \cos \alpha \mathcal{F}_{ab}'^2 + \sin \alpha \mathcal{F}_{ab}'^3 , \quad \mathcal{F}_{ab}'^3 \rightarrow -\sin \alpha \mathcal{F}_{ab}'^2 + \cos \alpha \mathcal{F}_{ab}'^3 \quad (19)$$

(b) The transformations associated with the generators  $T_2 \equiv -P$ ,  $T_3 \equiv -Q$ :

$$\mathcal{A}_a'^1 \rightarrow \mathcal{A}_a'^1 , \quad \mathcal{A}_a'^2 \rightarrow \mathcal{A}_a'^2 - \partial_a c_2 - \mathcal{A}_a'^1 c_3 , \quad \mathcal{A}_a'^3 \rightarrow \mathcal{A}_a'^3 - \partial_a c_3 + \mathcal{A}_a'^1 c_2 \quad (20)$$

where  $c_1$  and  $c_2$  are two local transformation parameters. The field strength components change as:

$$\mathcal{F}_{ab}'^1 \rightarrow \mathcal{F}_{ab}'^1 , \quad \mathcal{F}_{ab}'^2 \rightarrow \mathcal{F}_{ab}'^2 - \mathcal{F}_{ab}'^1 c_3 , \quad \mathcal{F}_{ab}'^3 \rightarrow \mathcal{F}_{ab}'^3 + \mathcal{F}_{ab}'^1 c_2 \quad (21)$$

Now, the first two equations of motion of  $ISO(2)$  Chern-Simons theory (17) are satisfied by the configurations where  $\mathcal{A}_v^i$  are pure gauge:

$$\begin{aligned}\mathcal{A}_v'^1 &= 0 , \quad \mathcal{A}_v'^2 = -\partial_v c_2 , \quad \mathcal{A}_v'^3 = -\partial_v c_3 , \\ \mathcal{A}_{\hat{a}}'^1 &= \mathcal{B}_{\hat{a}}'^1 , \quad \mathcal{A}_{\hat{a}}'^2 = \mathcal{B}_{\hat{a}}'^2 - \partial_{\hat{a}} c_2 - \mathcal{B}_{\hat{a}}'^1 c_3 , \quad \mathcal{A}_{\hat{a}}'^3 = \mathcal{B}_{\hat{a}}'^3 - \partial_{\hat{a}} c_3 + \mathcal{B}_{\hat{a}}'^1 c_2\end{aligned}\quad (22)$$

where  $\hat{a} = (\theta, \phi)$  and  $\mathcal{B}'_{\hat{a}}{}^i$  are independent of the coordinate  $v$ . For these most general configurations  $\mathcal{F}'_{v\theta}(\mathcal{A}') = 0$  and  $\mathcal{F}'_{v\phi}(\mathcal{A}') = 0$  hold identically and

$$\begin{aligned}\mathcal{F}'_{\theta\phi}(\mathcal{A}') &= \mathcal{F}'_{\theta\phi}(\mathcal{B}'), & \mathcal{F}'_{\theta\phi}(\mathcal{A}') &= \mathcal{F}'_{\theta\phi}(\mathcal{B}') - \mathcal{F}'_{\theta\phi}(\mathcal{B}')c_3, \\ \mathcal{F}'_{\theta\phi}(\mathcal{A}') &= \mathcal{F}'_{\theta\phi}(\mathcal{B}') + \mathcal{F}'_{\theta\phi}(\mathcal{B}')c_2\end{aligned}\quad (23)$$

From (17), the field strength components  $\mathcal{F}'_{\theta\phi}(\mathcal{B}')$  satisfy the equations of motion:

$$\frac{k}{2\pi} \mathcal{F}'_{\theta\phi}(\mathcal{B}') = \tilde{J}^i \quad \text{where} \quad \tilde{J}^1 = J^1, \quad \tilde{J}^2 = J^2 + c_3 J^1, \quad \tilde{J}^3 = J^3 - c_2 J^1 \quad (24)$$

For these equations, we are now left with invariance under  $v$ -independent  $ISO(2)$  gauge transformations. We use this freedom to make a gauge transformation of the type (b) above, by choosing the transformation parameters  $c_2(\theta, \phi)$  and  $c_3(\theta, \phi)$  appropriately:  $\mathcal{B}'_{\hat{a}} \rightarrow \mathcal{A}_{\hat{a}}^i$  and  $\mathcal{F}'_{\theta\phi}(\mathcal{B}') \rightarrow \mathcal{F}_{\theta\phi}^i(\mathcal{A})$  such that  $\mathcal{F}_{\theta\phi}^1(\mathcal{A}) \neq 0$ ,  $\mathcal{F}_{\theta\phi}^2(\mathcal{A}) = 0$  and  $\mathcal{F}_{\theta\phi}^3(\mathcal{A}) = 0$ . Consistent with this, the sources in (24) transform as:  $\tilde{J}^i \rightarrow J^i$  where  $J^i = (J, 0, 0)$  and finally the equation of motion (24) lead to:

$$\frac{k}{2\pi} \mathcal{F}_{\theta\phi}^1(\mathcal{A}) = J, \quad \mathcal{F}_{\theta\phi}^2(\mathcal{A}) = 0, \quad \mathcal{F}_{\theta\phi}^3(\mathcal{A}) = 0 \quad (25)$$

which, for  $k = \frac{\pi r_0^2}{\gamma}$  and  $J = \Sigma_{\theta\phi}^{(1)}$ , are same as (16). These equations are invariant under the left over  $U(1)$  transformations (type (a)) above. Thus we have demonstrated that the equations (16) are a partially gauge fixed version of the  $ISO(2)$  Chern-Simons equations (17) with a left over invariance only under  $U(1)$  transformations.

#### IV. $SU(2)$ CHERN-SIMONS BOUNDARY THEORY

Now we shall discuss that the horizon degrees of freedom can as well be described by a Chern-Simons theory of Barbero-Immirzi  $SU(2)$  gauge fields. To see this, we notice that the  $SU(2)$  gauge fields (9) on  $\Delta$  are:

$$\begin{aligned}A_a^{(1)} &\hat{=} \frac{\gamma}{2} \partial_a \ln \beta + \cos \theta \partial_a \phi, & A_a^{(2)} &\hat{=} -\sqrt{\beta} (\gamma \partial_a \theta - \sin \theta \partial_a \phi), \\ A_a^{(3)} &\hat{=} -\sqrt{\beta} (\gamma \sin \theta \partial_a \phi + \partial_a \theta)\end{aligned}\quad (26)$$

and the field strength components satisfy the following relations on  $\Delta$ :

$$\begin{aligned}F_{ab}^{(1)} &= 2 \partial_{[a} A_{b]}^{(1)} + 2 A_{[a}^{(2)} A_{b]}^{(3)} \hat{=} -\frac{2}{r_0^2} (1 - K^2) \Sigma_{ab}^{23} \\ F_{ab}^{(2)} &= 2 \partial_{[a} A_{b]}^{(2)} + 2 A_{[a}^{(3)} A_{b]}^{(1)} \hat{=} -2\sqrt{1 + \gamma^2} \sin \theta \partial_{[a} \phi \partial_{b]} K \\ F_{ab}^{(3)} &= 2 \partial_{[a} A_{b]}^{(3)} + 2 A_{[a}^{(1)} A_{b]}^{(2)} \hat{=} 2\sqrt{1 + \gamma^2} \partial_{[a} \theta \partial_{b]} K\end{aligned}\quad (27)$$

where  $K = \sqrt{\beta(1+\gamma^2)}$  which is arbitrary through spacetime dependent field  $\beta$  which can be changed by a boost transformation of the original tetrad and connection fields. Thus we may gauge fix this invariance under boost transformations by a convenient choice of  $\beta$  as follows:

(i) Now,  $\beta \equiv \frac{\alpha^2}{2e} = \frac{-vw}{2e} \hat{=} 0$  ( $K \hat{=} 0$ ) is a possible choice of the basis where the  $SU(2)$  gauge fields from (26) are:

$$A_a^{(1)} \hat{=} \frac{\gamma}{2} \partial_a \ln v + \cos \theta \partial_a \phi, \quad A_a^{(2)} \hat{=} 0, \quad A_a^{(3)} \hat{=} 0 \quad (28)$$

and from (27), the field strength components satisfy the following relations:

$$\begin{aligned} F_{ab}^{(1)} &= 2 \partial_{[a} A_{b]}^{(1)} + 2 A_{[a}^{(2)} A_{b]}^{(3)} \hat{=} - \frac{2}{r_0^2} \Sigma_{ab}^{23} = - \frac{2\gamma}{r_0^2} \Sigma_{ab}^{(1)} \\ F_{ab}^{(2)} &= 2 \partial_{[a} A_{b]}^{(2)} + 2 A_{[a}^{(3)} A_{b]}^{(1)} \hat{=} 0 \\ F_{ab}^{(3)} &= 2 \partial_{[a} A_{b]}^{(3)} + 2 A_{[a}^{(1)} A_{b]}^{(2)} \hat{=} 0 \end{aligned} \quad (29)$$

Notice that these equations are unaltered under the  $U(1)$  transformations:  $A_a^{(1)} \rightarrow A_a^{(1)} - \partial_a \xi$ ,  $A_a^{(2)} \rightarrow \cos \xi A_a^{(2)} + \sin \xi A_a^{(3)}$  and  $A_a^{(3)} \rightarrow -\sin \xi A_a^{(2)} + \cos \xi A_a^{(3)}$ . Hence, these equations can be interpreted as the equations of motion of a  $SU(2)$  Chern-Simons theory gauge fixed to a  $U(1)$  theory described by the  $U(1)$  gauge field  $A_a^{(1)}$  with coupling  $k = \frac{\pi r_0^2}{\gamma}$  and source  $\Sigma_{\theta\phi}^{(1)} \equiv \gamma^{-1} \Sigma_{\theta\phi}^{23}$  in the  $U(1)$  direction. An important property to note is that here  $\Sigma_{\theta\phi}^{(2)} \equiv \gamma^{-1} \Sigma_{\theta\phi}^{31} \hat{=} 0$  and  $\Sigma_{\theta\phi}^{(3)} \equiv \gamma^{-1} \Sigma_{\theta\phi}^{12} \hat{=} 0$ .

(ii) Other possible choice of the basis is where  $\beta$  is constant ( $K$  constant), but arbitrary. Here the gauge fields are:

$$\begin{aligned} A_a^{(1)} &\hat{=} \cos \theta \partial_a \phi, \quad A_a^{(2)} \hat{=} -K (\cos \delta \partial_a \theta - \sin \delta \sin \theta \partial_a \phi), \\ A_a^{(3)} &\hat{=} -K (\sin \delta \partial_a \theta + \cos \delta \sin \theta \partial_a \phi) \end{aligned} \quad (30)$$

where  $K = \sqrt{\beta(1+\gamma^2)}$  is now a constant and  $\cot \delta = \gamma$ . The right hand sides of last two equations in (27) are zero, that is, the field strength components satisfy:

$$\begin{aligned} F_{ab}^{(1)} &= 2 \partial_{[a} A_{b]}^{(1)} + 2 A_{[a}^{(2)} A_{b]}^{(3)} \hat{=} - \frac{2\gamma}{r_0^2} [1 - \beta(1+\gamma^2)] \Sigma_{ab}^{(1)} \\ F_{ab}^{(2)} &= 2 \partial_{[a} A_{b]}^{(2)} + 2 A_{[a}^{(3)} A_{b]}^{(1)} \hat{=} 0 \\ F_{ab}^{(3)} &= 2 \partial_{[a} A_{b]}^{(3)} + 2 A_{[a}^{(1)} A_{b]}^{(2)} \hat{=} 0 \end{aligned} \quad (31)$$

In these equations, we may interpret the combination

$$k = \frac{\pi r_0^2}{\gamma} \equiv \frac{a_H}{4\gamma}, \quad a_H \equiv 4\pi r_0^2 \quad (32)$$

as the  $SU(2)$  Chern-Simons coupling constant and the source as

$$J^{(i)} = \left( [1 - \beta (1 + \gamma^2)] \Sigma_{\theta\phi}^{(1)}, \quad 0, \quad 0 \right) \quad (33)$$

There is an arbitrary constant parameter  $\beta$  in the source which can be changed by a boost transformation of the original spin connection fields. Notice that for  $\beta = (1 + \gamma^2)^{-1}$ , the source vanishes.

Alternatively, we may take the combination  $k = \frac{a_H}{4\gamma[1-\beta(1+\gamma^2)]}$  as the coupling constant of the  $SU(2)$  Chern-Simons theory and  $J^{(i)} = \left( \Sigma_{\theta\phi}^{(1)}, \quad 0, \quad 0 \right)$  as the source in the  $U(1)$  direction of this theory. Then, we have a gauge dependent arbitrariness in the coupling constant, reflected through the parameter  $\beta$ . By boost transformations of the original spin connection fields, the value of  $\beta$  can be changed. For the specific choice  $\beta = \frac{1}{2}$ , we have the case of [15]. Also for  $\beta = (1 + \gamma^2)^{-1}$ , the coupling constant diverges. The ambiguity in how we define the Chern-Simons coupling strength depends on how we define bulk sources for the horizon Chern-Simons theory, which in turn depends on our choice of Lorentz frame in the bulk used to define the Schwarzschild spacetime in terms of tetrad frame components.

Like the equations of motion (29), eqns. (31) have invariance under the left over  $U(1)$  transformations. Thus, the gauge theory described by Eqns. (28)-(33), can be viewed as a  $SU(2)$  Chern-Simons theory on the horizon  $\Delta$  with a specific set of sources partially gauge fixed to  $U(1)$ . To see this explicitly, consider the  $SU(2)$  Chern-Simons theory with coupling  $k$  described by the action:

$$S_{CS} = \frac{k}{4\pi} \int_{\Delta} \epsilon^{abc} \left( A_a'^{(i)} \partial_b A_c'^{(i)} + \frac{1}{3} \epsilon^{ijk} A_a'^{(i)} A_b'^{(j)} A_c'^{(k)} \right) + \int_{\Delta} J'^{(i)a} A_a'^{(i)} \quad (34)$$

Here the nonzero components of the completely antisymmetric  $\epsilon^{abc}$  are given by  $\epsilon^{v\theta\phi} = 1$  and the source, which is a vector density with upper index  $a$ , is covariantly conserved,  $D_a(A') J'^{(i)a} \equiv \partial_a J'^{(i)a} + \epsilon^{ijk} A_a'^{(j)} J'^{(k)a} = 0$ , and further has the special form as:

$$J'^{(i)a} \equiv (J'^{(i)v}, J'^{(i)\theta}, J'^{(i)\phi}) = (J'^{(i)}, 0, 0) \quad (35)$$

The action (34) is independent of the metric of the three-manifold  $\Delta$ .

Now the equations of motion for the  $SU(2)$  Chern-Simons action (34) are:

$$F_{v\theta}'^{(i)}(A') \hat{=} 0, \quad F_{v\phi}'^{(i)}(A') \hat{=} 0, \quad \frac{k}{2\pi} F_{\theta\phi}'^{(i)}(A') \hat{=} - J'^{(i)} \quad (36)$$

The most general solution of the first two equations in this set is provided by the configurations where  $A_v'^{(i)}$  are pure gauge:

$$A_v'^{(i)} = -\frac{1}{2} \epsilon^{ijk} (\mathcal{O} \partial_v \mathcal{O}^T)^{jk}, \quad A_{\hat{a}}'^{(i)} = \mathcal{O}^{ij} B_{\hat{a}}'^{(j)} - \frac{1}{2} \epsilon^{ijk} (\mathcal{O} \partial_{\hat{a}} \mathcal{O}^T)^{jk}, \quad \hat{a} = (\theta, \phi) \quad (37)$$

with the  $SU(2)$  gauge fields  $B_{\theta}'^{(i)}$  and  $B_{\phi}'^{(i)}$  independent of  $v$ . Here  $\mathcal{O}$  is an arbitrary  $3 \times 3$  orthogonal matrix,  $\mathcal{O} \mathcal{O}^T = \mathcal{O}^T \mathcal{O} = 1$  with  $\det \mathcal{O} = 1$ . As  $F_{v\theta}'^{(i)}(A') \hat{=} 0$  and  $F_{v\phi}'^{(i)}(A') \hat{=} 0$  are identically satisfied, from (36), we are left with the equation:

$$\frac{k}{2\pi} F_{\theta\phi}'^{(i)}(A') = \frac{k}{2\pi} \mathcal{O}^{ij} F_{\theta\phi}'^{(j)}(B') \hat{=} -J'^{(i)} \quad (38)$$

where  $F_{\theta\phi}'^{(i)}(B')$  is the  $SU(2)$  field strength for the gauge fields  $(B_{\theta}'^{(i)}, B_{\phi}'^{(i)})$ .

This solution (37) has fixed part of the  $SU(2)$  gauge invariance; for the fields  $B_{\theta}'^{(i)}(\theta, \phi)$  and  $B_{\phi}'^{(i)}(\theta, \phi)$ , we are now left with invariance only under  $v$ -independent  $SU(2)$  gauge transformations on the spatial slice  $S^2$  of  $\Delta$ . Using this freedom, through a  $v$ -independent transformation matrix  $\bar{\mathcal{O}}(\theta, \phi)$ , it is always possible to write the triplet of field strength  $F_{\theta\phi}'^{(i)}(B')$  in terms of a field strength which is parallel to a unit vector  $u^i(\theta, \phi)$  in the internal space:

$$\begin{aligned} F_{\theta\phi}'^{(i)}(B') &= \bar{\mathcal{O}}^{ij} F_{\theta\phi}^{(j)}(B) \equiv u^i(\theta, \phi) F_{\theta\phi}^{(1)}(B), \\ F_{\theta\phi}^{(1)}(B) &\neq 0, \quad F_{\theta\phi}^{(2)}(B) = 0, \quad F_{\theta\phi}^{(3)}(B) = 0 \end{aligned} \quad (39)$$

where  $u^i(\theta, \phi) \equiv \bar{\mathcal{O}}^{i1}(\theta, \phi)$  and the gauge fields  $B_{\hat{a}}'^{(i)}(\theta, \phi)$  and  $B_{\hat{a}}^{(i)}(\theta, \phi)$  with the index  $\hat{a} = (\theta, \phi)$  are related by a gauge transformation as:

$$B_{\hat{a}}'^{(i)} = \bar{\mathcal{O}}^{ij} B_{\hat{a}}^{(j)} - \frac{1}{2} \epsilon^{ijk} (\bar{\mathcal{O}} \partial_{\hat{a}} \bar{\mathcal{O}})^{jk} \quad (40)$$

As discussed in the Appendix, there are two types of gauge fields  $B_{\hat{a}}^{(i)}(\theta, \phi)$  that yield the field strength, as in (39), parallel to the unit vector  $u^i$  which we may parameterized as  $u^i(\theta, \phi) = (\cos \Theta, \sin \Theta \cos \Phi, \sin \Theta \sin \Phi)$  in terms of two angles  $\Theta(\theta, \phi)$  and  $\Phi(\theta, \phi)$ . These two types are, from (A.10) and (A.15): (i)  $B_{\hat{a}}^{(i)} = (B_{\hat{a}} + \cos \Theta \partial_{\hat{a}} \Phi, 0, 0)$  with  $B_{\hat{a}}$  as arbitrary. This corresponds to the configuration (28) with its field strength as in (29) above for  $B_{\hat{a}} = 0$  and  $\Theta = \theta, \Phi = \phi$ . (ii) The second solution is:  $B_{\hat{a}}^{(1)} = -\partial_{\hat{a}} \delta + \cos \Theta \partial_{\hat{a}} \Phi$ ,  $B_{\hat{a}}^{(2)} = c(\cos \delta \partial_{\hat{a}} \Theta - \sin \delta \sin \Theta \partial_{\hat{a}} \Phi)$ ,  $B_{\hat{a}}^{(3)} = c(\sin \delta \partial_{\hat{a}} \Theta + \cos \delta \sin \Theta \partial_{\hat{a}} \Phi)$  where  $c$  is a constant and  $\delta(\theta, \phi)$  is arbitrary. Notice that this configuration is the same as that describing the horizon fields in (30) with  $c = -K$  and  $\Theta\theta, \Phi = \phi$  and  $\delta$  as constant. The corresponding

field strength components satisfy the equations of motion given by (31) with the coupling and the sources as identified by (32) and (33).

Thus we may rewrite the equations (37), for both these cases, as:

$$A_v'^{(i)} = -\frac{1}{2} \epsilon^{ijk} \left( \mathcal{O}' \partial_v \mathcal{O}'^T \right)^{jk}, \quad A_a'^{(i)} = \mathcal{O}'^{ij} B_a^{(j)} - \frac{1}{2} \epsilon^{ijk} \left( \mathcal{O}' \partial_a \mathcal{O}'^T \right)^{jk} \quad (41)$$

where  $\mathcal{O}' \mathcal{O} \bar{\mathcal{O}}$ . The field strength components  $F_{v\theta}'^{(i)}(A')$  and  $F_{v\phi}'^{(i)}(A')$  are identically zero and the equation (38) becomes

$$\frac{k}{2\pi} F_{\theta\phi}'^{(i)}(A') = \frac{k}{2\pi} \mathcal{O}'^{ij} F_{\theta\phi}^{(j)}(B) \hat{=} - J^{(i)} \equiv - \mathcal{O}'^{ij} J^{(j)} \quad (42)$$

where now from (39),  $F_{\theta\phi}^{(i)}(B) = \left( F_{\theta\phi}^{(1)}(B), 0, 0 \right)$ , which implies for the sources

$$J^{(i)} = (J, 0, 0) \quad (43)$$

As discussed in the Appendix, in terms of the fields  $B_a^{(i)}$  and the corresponding field strength  $\left( F_{\theta\phi}^{(1)}(B), 0, 0 \right)$ , we have a theory with left over invariance only under  $U(1)$  gauge transformations. Thus, in the  $SU(2)$  theory partially gauge fixed to a theory with invariance only under  $U(1)$  transformations, *the sources in the internal space directions orthogonal to the  $U(1)$  are zero; the only source is in the direction of the  $U(1)$  subgroup*. Further, the coupling constant of the  $SU(2)$  Chern-Simons theory is given by  $k = \frac{\pi r_0^2}{\gamma}$ .

## V. CONCLUDING REMARKS

That there are different, but equivalent, classical formulations of the topological theory of the horizon degrees of freedom is to do with the fact that it is essentially only the properties of various fields on the spatial slice  $S^2$  of the horizon that are relevant. Note in this respect that our approach is quite complementary to the Hamiltonian analysis of isolated horizons [1], [15]. Though our analysis here has been restricted to the case of the event horizon of the Schwarzschild solution, many of our conclusions do in fact generalize for generic spherical isolated horizons. However, a Hamiltonian analysis of the constraints of the theory in presence of isolated horizons described by a set of boundary conditions, as has been done in the quoted references, could be performed. Classical Hamiltonian formulation of the  $SU(2)$  Chern-Simons theory on the event horizon has three first class constraints corresponding to the three generators of  $SU(2)$  gauge transformations. On three-manifolds

with topology of  $S^2 \times R$ , in the process of gauge fixing from  $SU(2)$  to  $U(1)$ , two of these are gauge fixed through gauge fixing constraints with which these form a set of second class constraints according to the standard rules of gauge fixing. To implement these second class constraints, we need to go over from the Poisson brackets to the corresponding Dirac brackets. We are then left with only one first class constraint associated with the left over  $U(1)$  invariance.

In Loop Quantum Gravity where the bulk properties are described by the quantum theory based on Barbero-Immirzi  $SU(2)$  gauge theory, the horizon degrees of freedom are described by an  $SU(2)$  Chern-Simons theory, or equivalently its gauge fixed version in terms of a theory exhibiting only a left over  $U(1)$  invariance but with additional constraints on the solder forms. Further, there are no local degrees of freedom in topological quantum Chern-Simons theories; all the degrees of freedom are global or topological. These global degrees of freedom reside in the properties of the punctures on  $S^2$ . These are given by the spin networks from the bulk quantum theory where we have  $SU(2)$  spins living on these punctures. This information is in the values of the solder forms on  $S^2$  which, in the quantum theory, have distributional support at these punctures.

Properties of the quantum black holes can be calculated in either formulation,  $SU(2)$  or the partially gauge fixed version with only the left over  $U(1)$  invariance, yielding the same results. In particular, the black hole entropy in either formulation has the standard leading area law and the subleading correction given by logarithm of area with definite coefficient  $-3/2$  for large black hole area as obtained in [10, 11]. The value of the Barbero-Immirzi parameter  $\gamma$  obtained by matching the leading area term with the Bekenstein-Hawking law is also the same. However, as already mentioned and also emphasized earlier in [13], care needs to be exercised in doing the calculations in the  $U(1)$  formulation by implementing the extra conditions on the solder forms on the quantum states contributing to the entropy.

Though, in the realistic situation of a sufficiently massive star collapsing gravitationally, the past horizon ( $v = 0$ ) of the idealized Kruskal-Szekeres extended Schwarzschild geometry is never realized, it is of interest to note that the discussion developed above holds for this horizon also. Its degrees of freedom are again described by an  $SU(2)$  Chern-Simons theory, or equivalently its gauge fixed version in terms of a  $U(1)$  theory.



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## Appendix: $U(1)$ Gauge theory as gauge fixed $SU(2)$ Chern-Simons on $S^2$

Here we explicitly discuss how the  $SU(2)$  fields  $B_{\hat{a}}^{(i)}(\theta, \phi)$  of Eqn. (40) can be gauge fixed to the fields  $B_{\hat{a}}^{(i)}$  with only a left over  $U(1)$  invariance on the spatial slice  $S^2$  of the horizon. To unravel the nature of the fields  $B_{\hat{a}}^{(i)}(\theta, \phi)$ , we may parametrize these along the internal space unit vector  $u^i(\theta, \phi)$  and orthogonal to it as:

$$B_{\hat{a}}^{(i)} = u^i B_{\hat{a}} + f \partial_{\hat{a}} u^i + g \epsilon^{ijk} u^j \partial_{\hat{a}} u^k, \quad \hat{a} = (\theta, \phi) \quad (\text{A.1})$$

where  $f$  and  $g$  are functions on the spacetime  $S^2$ . Then the six independent field degrees of freedom in  $B_{\hat{a}}^{(i)}$  are now distributed in  $u^i$  (two independent fields),  $B_{\hat{a}}$  (two field degrees of freedom) and the two fields  $(f, g)$ . Had the internal space unit vector  $u^i$  in (A.1) been completely arbitrary, all these six field degrees of freedom would be independent. Here  $u^i$  is, through (39), parallel to the field strength  $F_{\theta\phi}^{(i)}(B')$ . For this reason, all of these six field degrees of freedom are not independent.

Now the field strength for the gauge fields (A.1) can be calculated in a straight forward manner to be:

$$\begin{aligned} F_{\hat{a}\hat{b}}^{(i)}(B') &= u^i \left( 2 \partial_{[\hat{a}} B_{\hat{b}]} + (f^2 + g^2 + 2g) \epsilon^{ijk} u^j \partial_{\hat{a}} u^k \partial_{\hat{b}} u^l \right) \\ &+ 2 \partial_{[\hat{a}} u^i \left( (1+g) B_{\hat{b}]} - \partial_{\hat{b}]} f \right) - 2 \epsilon^{ijk} u^j \partial_{[\hat{a}} u^k \left( f B_{\hat{b}]} + \partial_{\hat{b}]} g \right) \end{aligned} \quad (\text{A.2})$$

where we have used the identity for the internal space unit vector  $u^i$ :

$$\epsilon^{ijk} \partial_{\hat{a}} u^j \partial_{\hat{b}} u^k = u^i \epsilon^{jkl} u^j \partial_{\hat{a}} u^k \partial_{\hat{b}} u^l \quad (\text{A.3})$$

Now comparing (A.2) with (39) where the components orthogonal to  $u^i$  are zero, we have the conditions:

$$\partial_{\hat{a}} f - (1+g) B_{\hat{a}} = 0, \quad \partial_{\hat{a}} g + f B_{\hat{a}} = 0 \quad (\text{A.4})$$

These conditions imply that there are two classes of the  $SU(2)$  gauge fields  $B_{\hat{a}}'^{(i)}$  on the spatial slice  $S^2$  which give such a field strength:

(i) The first kind are where the field  $B_{\hat{a}}$  is arbitrary and  $f = 0$  and  $1 + g = 0$ . Then the gauge field from (A.1) is:

$$B_{\hat{a}}'^{(i)} = u^i B_{\hat{a}} - \epsilon^{ijk} u^j \partial_{\hat{a}} u^k \quad (\text{A.5})$$

Clearly the field strength for such a gauge field is parallel to  $u^i$ :

$$F_{\hat{a}\hat{b}}'^{(i)}(B') = u^i \left( 2 \partial_{[\hat{a}} B_{\hat{b}]} - \epsilon^{jkl} u^j \partial_{\hat{a}} u^k \partial_{\hat{b}} u^l \right) \quad (\text{A.6})$$

The quantity  $\epsilon^{\hat{a}\hat{b}} \epsilon^{jkl} u^j \partial_{\hat{a}} u^k \partial_{\hat{b}} u^l$  is the *winding number* density for the homotopy maps  $S^2 \rightarrow S^2$  and its integral over the two-dimensional space  $S^2$  is characterized by integers (Homotopy group  $\Pi_2(S^2) = \mathbb{Z}$ ). Since it is a topological density we can write it as

$$\epsilon^{jkl} u^j \partial_{\hat{a}} u^k \partial_{\hat{b}} u^l = 2 \partial_{[\hat{a}} \Omega_{\hat{b}]} \quad (\text{A.7})$$

In particular, for the parameterization of unit vector  $u^i$  in terms two angles  $\Theta(\theta, \phi)$  and  $\Phi(\theta, \phi)$  as  $u^i = (\cos \Theta, \sin \Theta \cos \Phi, \sin \Theta \sin \Phi)$ , we have,

$$\epsilon^{jkl} u^j \partial_{\hat{a}} u^k \partial_{\hat{b}} u^l = 2 \sin \Theta \partial_{[\hat{a}} \partial_{\hat{b}]} \Phi \quad \text{and} \quad \Omega_{\hat{a}} = -\cos \Theta \partial_{\hat{a}} \Phi. \quad (\text{A.8})$$

Further, for the unimodular orthogonal matrix  $\bar{\mathcal{O}}^{ij}$  with its components as:  $\bar{\mathcal{O}}^{i1} = u^i = (\cos \Theta, \sin \Theta \cos \Phi, \sin \Theta \sin \Phi)$ ,  $\bar{\mathcal{O}}^{i2} = (-\sin \Theta, \cos \Theta \cos \Phi, \cos \Theta \sin \Phi)$ ,  $\bar{\mathcal{O}}^{i3} = (0, -\sin \Phi, \cos \Phi)$ , the following identity can be shown to hold:

$$\epsilon^{ijk} u^j \partial_{\hat{a}} u^k = u^i \Omega_{\hat{a}} + \frac{1}{2} \epsilon^{ijk} \bar{\mathcal{O}}^{jl} \partial_{\hat{a}} \bar{\mathcal{O}}^{kl} \quad (\text{A.9})$$

Defining  $\mathcal{B}_{\hat{a}} \equiv B_{\hat{a}} - \Omega_{\hat{a}}$ , this identity allows us to rewrite the gauge field (A.5) and its field strength (A.6) as:

$$\begin{aligned} B_{\hat{a}}'^{(i)} &= u^i \mathcal{B}_{\hat{a}} - \frac{1}{2} \epsilon^{ijk} (\bar{\mathcal{O}} \partial_{\hat{a}} \bar{\mathcal{O}}^T)^{jk} \equiv \bar{\mathcal{O}}^{ij} B_{\hat{a}}^{(j)} - \frac{1}{2} \epsilon^{ijk} (\bar{\mathcal{O}} \partial_{\hat{a}} \bar{\mathcal{O}}^T)^{jk}, \\ F_{\hat{a}\hat{b}}'^{(i)}(B') &= 2 u^i \partial_{[\hat{a}} \mathcal{B}_{\hat{b}]} \equiv 2 \bar{\mathcal{O}}^{ij} \partial_{[\hat{a}} B_{\hat{b}]}^{(j)} \end{aligned} \quad (\text{A.10})$$

with the  $SU(2)$  gauge field  $B_{\hat{a}}^{(i)} \equiv (B_{\hat{a}}^{(1)}, B_{\hat{a}}^{(2)}, B_{\hat{a}}^{(3)}) = (\mathcal{B}_{\hat{a}}, 0, 0)$ . Thus, the fields  $\mathcal{B}_{\hat{a}}$  and their field strength  $\mathcal{F}_{\hat{a}\hat{b}} = 2 \partial_{[\hat{a}} \mathcal{B}_{\hat{b}]}$ , describe  $U(1)$  gauge configurations obtained from  $SU(2)$  gauge fields by partial gauge fixing.

(ii) The second class of gauge fields are where  $B_{\hat{a}} = -\partial_{\hat{a}}\delta$ ,  $f = c \cos \delta$  and  $1+g = c \sin \delta$  with  $c$  as a constant and  $\delta(\theta, \phi)$  an arbitrary field. For these values we have,  $f^2 + (1+g)^2 = c^2$ . Thus, the gauge fields from (A.1) are:

$$B_{\hat{a}}'^{(i)} = -u^i \partial_{\hat{a}}\delta + c \cos \delta \partial_{\hat{a}}u^i + (c \sin \delta - 1) \epsilon^{ijk} u^j \partial_{\hat{a}}u^k \quad (\text{A.11})$$

For these gauge fields, the field strength is again parallel to  $u^i(\theta, \phi)$ :

$$F_{\hat{a}\hat{b}}'^{(i)}(B') = (c^2 - 1) u^i \epsilon^{jkl} u^j \partial_{\hat{a}}u^k \partial_{\hat{b}}u^l \quad (\text{A.12})$$

which can again be rewritten as:

$$F_{\hat{a}\hat{b}}'^{(i)}(B') = u^i \mathcal{F}_{\hat{a}\hat{b}}, \quad \mathcal{F}_{\hat{a}\hat{b}} = 2 \partial_{[\hat{a}} \mathcal{B}_{\hat{b}]} \quad \text{with} \quad \mathcal{B}_{\hat{a}} = c^2 \Omega_{\hat{a}} - \Omega_{\hat{a}} \quad (\text{A.13})$$

where  $\Omega_{\hat{a}}$  is given by (A.7). This field strength then is completely characterized by the  $U(1)$  theory of gauge field  $\mathcal{B}_{\hat{a}}$  and its field strength  $\mathcal{F}_{\hat{a}\hat{b}}$ .

Using the parametrization for the unit vector  $u^i$  as earlier and the identity (A.9), the gauge fields (A.11) can be rewritten as:

$$B_{\hat{a}}'^{(i)} = \bar{\mathcal{O}}^{ij} B_{\hat{a}}^{(j)} - \frac{1}{2} \epsilon^{ijk} \bar{\mathcal{O}}^{jl} \partial_{\hat{a}} \bar{\mathcal{O}}^{kl} \quad (\text{A.14})$$

where, in this case the gauge field  $B_{\hat{a}}^{(i)}$  has all three internal space components non-zero, but of a specific form:

$$\begin{aligned} B_{\hat{a}}^{(1)} &= -\partial_{\hat{a}}\delta + \cos \Theta \partial_{\hat{a}}\Phi, \\ B_{\hat{a}}^{(2)} &= c (\cos \delta \partial_{\hat{a}}\Theta - \sin \delta \sin \Theta \partial_{\hat{a}}\Phi), \\ B_{\hat{a}}^{(3)} &= c (\sin \delta \partial_{\hat{a}}\Theta + \cos \delta \sin \Theta \partial_{\hat{a}}\Phi) \end{aligned} \quad (\text{A.15})$$

This partially gauge fixed  $SU(2)$  theory described by the gauge fields (A.10) and (A.15) with field strength (A.13), has only a left over gauge invariance under  $U(1)$  transformations:  $B_{\hat{a}}^{(1)} \rightarrow B_{\hat{a}}^{(1)} + \partial_{\hat{a}}\lambda$ ,  $B_{\hat{a}}^{(2)} \rightarrow \cos \lambda B_{\hat{a}}^{(2)} + \sin \lambda B_{\hat{a}}^{(3)}$ ,  $B_{\hat{a}}^{(3)} \rightarrow -\sin \lambda B_{\hat{a}}^{(2)} + \cos \lambda B_{\hat{a}}^{(3)}$ . We may use this invariance to rotate away the arbitrary field  $\delta$  to zero in (A.15) through an  $U(1)$  transformation with  $\lambda = \delta$ .

An interesting property to note is that the gauge configurations (A.5) of class (i) satisfy the condition that the unit vector  $u^i$  is covariantly constant, that is,  $D_{\hat{a}}(B')u^i \equiv \partial_{\hat{a}}u^i + \epsilon^{ijk} B_{\hat{a}}'^{(j)}u^k = 0$ . On the other hand, the configurations corresponding to the class

(ii) given by (A.11) do not satisfy this condition; instead these satisfy:  $D_{\hat{a}}(B')u^i = c(\sin \delta \partial_{\hat{a}} u^i - \cos \delta \epsilon^{ijk} u^j \partial_{\hat{a}} u^k) \neq 0$  for  $c \neq 0$ .

For  $c = 0$ , the configuration (A.15) of class (ii) reduces to (A.5) of the class (i) above with  $B_{\hat{a}} = 0$ . But for  $c \neq 0$ , the configurations of these two classes are completely distinct.

In fact there is a general underlying mathematical reason for the fact that  $SU(2)$  Chern-Simons theory on a manifold  $R \times S^2$  with punctures on  $S^2$ , like the horizon, can be described by a  $U(1)$  theory with consequent conditions on the special sources (35) of the  $SU(2)$  Chern-Simons theory that these are zero in the directions orthogonal to  $U(1)$  direction given by the internal space vector  $u^i$  as given in (43). This is as follows: In general, the gauge group  $SU(2)$ , with its group manifold being  $S^3$ , can be mapped to a  $S^2$  ( $SU(2)/U(1)$  coset space) in such a way that every point on  $S^2$  comes from a circle on  $S^3$ . Such maps, known as Hopf maps, generate the homotopy group  $\Pi_3(S^2)$  which is just the set of integers  $\mathbb{Z}$ . So  $SU(2)$  is a bundle (the Hopf bundle) over  $S^2$  with a  $U(1)$  fibre. Now, any  $SU(2)$  gauge field  $(B_{\theta}^{(i)}, B_{\phi}^{(i)})$  in two dimensional spacetime  $S^2$  can be, in general, gauge fixed in the internal space directions contained in the coset space  $SU(2)/U(1) \cong S^2$  (which are orthogonal to the internal space unit vector  $u^i$  characterizing this  $S^2$ ). This gauge fixing can be achieved through an appropriate gauge transformation from the coset space  $SU(2)/U(1)$ . Then we are just left with invariance under the  $U(1)$  transformations which leave this unit vector  $u^i$  unaltered. Thus an arbitrary  $SU(2)$  gauge field  $(B_{\theta}^{(i)}, B_{\phi}^{(i)})$  on spacetime  $S^2$  is completely characterized by the internal space unit vector  $u^i$  and the gauge field  $(\mathcal{B}_{\theta}, \mathcal{B}_{\phi})$  of the  $U(1)$  subgroup in the direction  $u^i$ . The corresponding  $SU(2)$  field strength is parallel to  $u^i$  and is completely determined by the  $U(1)$  field strength.

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